Comment on "How the potentials in different gauges yield the same retarded electric and magnetic fields," by J. A. Heras [Am. J. Phys. 75, 176 (2007)]

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In a recent paper,[1] Heras surveys the Lorenz, Coulomb, Kirchhoff, velocity, and temporal gauges with a view to explaining how the potentials of all these gauges yield the same retarded electromagnetic field, despite the fact that these potentials may satisfy dynamical equations that do not admit properly retarded solutions. He claims to show without actually solving the equations that the potentials satisfy that the "spurious" non-causal term generated by the scalar potential of the Coulomb, Kirchhoff, or velocity gauge is canceled by an equal and opposite term in the contribution to the electric field that is generated by the vector potential. With no intention of diminishing the value of the gauge survey itself, we argue that, given the definitions of the electric and magnetic fields in terms of electromagnetic potentials, Heras's cancelation of a non-causal term in the electric field is an artefact of algebraic manipulation that has no explanatory content.

Heras manipulates the equations that, for example, the potentials Φ_C and $\mathbf{A}_C$ of the Coulomb gauge satisfy in four distinct steps, which in the end lead to inhomogeneous wave equations

$$\Box^2\left(-\nabla\Phi_C-\frac{\partial\mathbf{A}_C}{\partial t}\right) = \frac{1}{\epsilon_0}\nabla\rho+\mu_0\frac{\partial\mathbf{J}}{\partial t}, \tag{1}$$

$$\Box^2(\nabla\times\mathbf{A}_C) = -\mu_0\nabla\times\mathbf{J}. \tag{2}$$

He then writes the retarded solution $-\nabla\Phi_C-\partial\mathbf{A}_C/\partial t$ of (1) as

$$-\frac{\partial\mathbf{A}_C}{\partial t}=\frac{1}{4\pi\epsilon_0}\int\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\left[-\nabla'\rho-\frac{1}{c^2}\frac{\partial\mathbf{J}}{\partial t'}\right]_{\rm ret}+\nabla\Phi_C, \tag{3}$$

which must give the electric field $\mathbf{E}=-\nabla\Phi_C-\partial\mathbf{A}_C/\partial t$ as

$$\mathbf{E}=\frac{1}{4\pi\epsilon_0}\int\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\left[-\nabla'\rho-\frac{1}{c^2}\frac{\partial\mathbf{J}}{\partial t'}\right]_{\rm ret} \tag{4}$$

since the instantaneous term $-\nabla\Phi_C$ in $-\nabla\Phi_C-\partial\mathbf{A}_C/\partial t$ is canceled by an equal and opposite term in $-\partial\mathbf{A}_C/\partial t$ of (3). Because of this cancelation, Heras calls the term $-\nabla\Phi_C$ a "spurious" field, which has only a "mathematical", but not "physical" existence.

We note first that Heras's four steps are not needed to obtain the wave equations (1) and (2) — Maxwell's equations lead directly, without the aid of any potentials, to inhomogeneous wave equations for the electric and magnetic fields themselves:[2]

$$\Box^2\mathbf{E} \;=\; \frac{1}{\epsilon_0}\nabla\rho + \mu_0\frac{\partial \mathbf{J}}{\partial t}, \tag{5}$$

$$\Box^2\mathbf{B} \;=\; -\mu_0\nabla\times\mathbf{J}, \tag{6}$$

the retarded solutions of which are

$$\mathbf{E} \;=\; \frac{1}{4\pi\epsilon_0}\int\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\left[-\nabla'\rho - \frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t'}\right]_{\rm ret}, \tag{7}$$

$$\mathbf{B} \;=\; \frac{\mu_0}{4\pi}\int\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}[\nabla'\times\mathbf{J}]_{\rm ret}. \tag{8}$$

Equations (1)–(3) then follow immediately on the use in Eqs. (5)–(7) of the definitions

$$\mathbf{E} \;=\; -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{9}$$

$$\mathbf{B} \;=\; \nabla\times\mathbf{A}, \tag{10}$$

where Φ and $\mathbf{A}$ are potentials in an arbitrary gauge. Heras has to employ the definitions (9) and (10) already when he derives from Maxwell's equations the dynamical equations for the potentials of the gauges he considers. The retarded solution (3), written in the usual manner, thus holds in an arbitrary gauge:

$$-\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} = \frac{1}{4\pi\epsilon_0}\int\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\left[-\nabla'\rho - \frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t'}\right]_{\rm ret}. \tag{11}$$

Equations (7)–(10) ensure that, irrespective of the gauge used, the retarded electromagnetic field is given uniquely by its sources ρ and $\mathbf{J}$, but do not help to answer the perhaps superfluous, but still intriguing, question *how* that can happen when the potentials themselves are solutions of dynamical equations that do not admit properly retarded solutions. Only constructive solutions of the dynamical equations for the potentials in terms of the sources ρ and $\mathbf{J}$ can help in this regard, like, for example, the solution obtained by Jackson[3] for the Coulomb-gauge vector potential $\mathbf{A}_C$,

$$\mathbf{A}_C = \frac{\mu_0}{4\pi}\int\frac{d^3x'}{R}\left([\mathbf{J} - c\rho\hat{\mathbf{R}}]_{\rm ret} + \frac{c^2\hat{\mathbf{R}}}{R}\int_0^{R/c} d\tau\,\rho(\mathbf{x}', t-\tau)\right), \tag{12}$$

where $R = |\mathbf{x}-\mathbf{x}'|$ and $\hat{\mathbf{R}} = (\mathbf{x}-\mathbf{x}')/R$. The partial time derivative of (12) yields

$$-\frac{\partial \mathbf{A}_C}{\partial t} = \frac{1}{4\pi\epsilon_0}\int d^3x'\left(\left[\frac{\rho}{R^2}\hat{\mathbf{R}} + \frac{1}{cR}\frac{\partial\rho}{\partial t'}\hat{\mathbf{R}} - \frac{1}{c^2R}\frac{\partial \mathbf{J}}{\partial t'}\right]_{\rm ret} - \frac{\rho}{R^2}\hat{\mathbf{R}}\right). \tag{13}$$

Here, the retarded terms in the integrand give Jefimenko's expression for the retarded electric field,[4] while the last term yields an instantaneous term $\nabla\Phi_C$, which will be canceled by the equal and opposite term in (9).

In contrast, it is not difficult to see that the cancelation of the term $-\nabla\Phi_C$ in $\mathbf{E}$ by an equal and opposite term in $-\partial\mathbf{A}_C/\partial t$ obtained by Heras is merely an artefact of writing, in Eq. (3), the retarded solution of the inhomogeneous wave equation (1), which is the retarded electric field $\mathbf{E}$ of Eq. (4), as $-\partial\mathbf{A}_C/\partial t = \mathbf{E} + \nabla\Phi_C$. Using the Lorenz-gauge potentials Φ_L and $\mathbf{A}_L$ in Eqs. (9) and (11), one could argue *à la* Heras that the retarded term $-\nabla\Phi_L$ in $\mathbf{E}$ is also a "spurious" field because it is canceled by an equal and opposite term in $-\partial\mathbf{A}_L/\partial t$, the arbitrary exclusion by Heras of the Lorenz gauge from the four steps of his method notwithstanding. In general, neither of the potential-generated terms $-\nabla\Phi$ and $-\partial\mathbf{A}/\partial t$ alone represents a physical field, but that does not mean that any of these terms can be regarded as spurious because both are needed in the sum $-\nabla\Phi - \partial\mathbf{A}/\partial t$ that is guaranteed by the mathematical consistency of the electromagnetic-potential method of solving Maxwell's equations to yield the physical electric field $\mathbf{E}$.

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[1]J. A. Heras, "How the potentials in different gauges yield the same retarded electric and magnetic fields," Am. J. Phys. **75**, 176–183 (2007).

[2]The inhomogeneous wave equations for **E** and **B** can be obtained by taking curls of Maxwell's curl equations and decoupling the resulting equations by using the remaining pair of Maxwell's equations.

[3]J. D. Jackson, "From Lorenz to Coulomb and other explicit gauge transformations," Am. J. Phys. **70**, 917–928 (2002).

[4]O. D. Jefimenko, *Electricity and Magnetism* (Electret Scientific, Star City, 1989), 2nd ed.; J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1999), 3rd ed., Sec. 6.5.